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Dirac Masses Determination with Orthogonal Polynomials and ε-Algorithm. Application to Totally Monotonic Sequences

MARC PRÉVOST

Université des Sciences et Techniques de Lille Flandres-Artois, Laboratoire d'Analyse Numérique et d'Optimisation, UFR d'IEEA-Bât. M3, 59655 Villeneuve d'Ascq-Cédex, France

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Let us consider a distribution c with is the sum of another distribution, ω, and a linear combination of Dirac distribution with masses $A_1, ..., A_q$ at the points $a_1, ..., a_q$. We have proposed a method (*in* "IMACS Transaction on Orthogonal Polynomials and Their Applications" (C. Brezinski, L. Gori, and A. Ronveaux, Eds.), pp. 365–372, Baltzer, Basel, 1991) to compute the points $a_1, ..., a_q$ when ω has an asymptotic property when applied to a given sequence of orthogonal polynomials. This paper is devoted to the computation of the masses $A_1, ..., A_q$ with the aid of Chebyshev polynomials, Christoffel function and ε-algorithm according to the distribution ω. An application to finding the limit of totally monotonic sequence is also given.

1. INTRODUCTION AND NOTATIONS

The space of linear functionals (distributions) defined on P, space of polynomials, will be denoted by P'. The Dirac distribution δ_{x_0} is defined as follows:

$$\langle \delta_{x_0}, p \rangle = p(x_0), \qquad p \in P, \qquad x_0 \in \mathbf{C}.$$

Let c and ω be two elements of P'. We assume that the moments

$$c_n := c(x^n) := \langle c, x^n \rangle, \qquad n = 0, 1, ...,$$

are given and that c and ω are related by the following relation:

$$c = \omega + \sum_{i=1}^{q} A_i \delta_{a_i} \qquad a_i \in [-1, 1], a_i \text{ distincts}, A_i \in \mathbf{C} - \{0\}. \qquad (1)$$

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If ω has an integral representation whose support is a compact interval of **R**, then a general convergence theorem of Goncar [10] for the Padé approximants to the sum of a Markov–Stieltjes function $h(t)$ and a rational fraction $r(t)$ permits to compute the points $a_1, ..., a_q$ (which are the inverse of the poles of $r(t)$) and the residues $A_1, ..., A_q$ in the case where $a_1, ..., a_q$ do not lie on the cut of $h(t)$.

If ω is absolutely continuous with respect to the Lebesgue measure, with a positive density w, then w can be approximated by methods explained in [15] and using Christoffel functions, with Turan determinants [15, p. 80; 1], or with continued fraction [2, 11]).

Here, we present a method for computing the residues $A_1, ..., A_q$ when points $a_1, ..., a_q$ are known (or computed by the method described in [16]) when ω is absolutely continuous (ω non necessarily positive) with respect to the Lebesgue measure on a compact interval of **R** (section 2) or when ω is a positive distribution on **R** (Section 3).

2. Chebyshev Polynomials

Let c and ω be two distributions defined on P and which satisfy relation (1).

Moreover, we suppose that ω has an integral representation on a compact interval of **R** which can be assumed to be $[-1, 1]$:

$$\langle \omega, p \rangle := \int_{-1}^{1} p(x)\, w(x)\, dx \qquad w \in L^1. \tag{2}$$

In order to get the mass A_l at the point a_l, the idea is to apply both sides of equality (1) to the characteristic function $\chi_{\{a_l\}}$ defined as the following:

$$\chi_{\{a_l\}}(x) = 1 \qquad \text{if} \quad x = a_l, \qquad 0 \text{ otherwise.}$$

This formally gives

$$\langle c, \chi_{\{a_l\}} \rangle = \langle \omega, \chi_{\{a_l\}} \rangle + A_l = A_l,$$

since ω is absolutely continuous with respect to dx. Thus, A_l appears as the moment of $\chi_{\{a_l\}}$ by the distribution c. Since only the polynomial moments c_n are known, it is realistic to approximate the characteristic function by a sequence of polynomials $l_n \in P_n$ in a way such that the equality

$$A_l = \lim_{n \to \infty} \langle c, l_n \rangle$$

holds.

A convenient means to construct such polynomials l_n is the kernel polynomial for the weight $dx/\sqrt{1-x^2}$. Let us first recall some properties of Chebyshev polynomials of the first kind T_n:

$$T_n(x) = \cos(n \operatorname{Arccos} x) \qquad n = 0, 1, \dots .$$

Three-terms recurrence relationship:

$$\left| \begin{array}{ll} T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) & n = 1, 2, \dots \\ T_0 = 1 \qquad T_1 = x. & \end{array} \right.$$

Orthogonality relation:

$$k \in \mathbf{N}, \qquad l \in \mathbf{N}^*, \qquad \frac{2}{\pi} \int_{-1}^{+1} T_k(x)\, T_l(x) \frac{dx}{\sqrt{1-x^2}} = \delta_{kl} \text{ (Kronecker symbol).}$$

The sequence $(1/\sqrt{\pi})\, T_0, \sqrt{(2/\pi)}\, T_1(x), \dots, \sqrt{(2/\pi)}\, T_n(x)$ is the system of orthonormal polynomials with respect to the weight function $1/\sqrt{1-x^2}$, $-1 < x < 1$.

Let us consider, now, the reproducing kernel polynomial (which is orthogonal w.r.t. $(x-a)\, dx/\sqrt{1-x^2}$)

$$k_n(x, a) = [T_{n+1}(x)\, T_n(a) - T_n(x)\, T_{n+1}(a)]/(x-a) \tag{3}$$

which, due to Christoffel identity can be rewritten as

$$k_n(x, a) = {\sum_{k=0}^{n}}' T_k(x)\, T_k(a),$$

where ${\sum'}_{i=0}^{n} u_i := \frac{1}{2}u_0 + u_1 + \cdots + u_n$.

From the definition of T_n, it follows immediately that

$$|(x-a)\, k_n(x, a)| \leqslant 2, \qquad -1 \leqslant x \leqslant +1, \qquad -1 \leqslant a \leqslant +1$$

The value of $k_n(x, a)$ at $x = a$ is

$$k_n(a, a) = {\sum_{k=0}^{n}}' T_k^2(a) = T'_{n+1}(a)\, T_n(a) - T'_n(a)\, T_{n+1}(a).$$

An identity between T_n and U_n, Chebyshev polynomial of second kind also gives

$$k_n(a, a) = \tfrac{1}{4}(U_{2n}(a) + 2n + 1).$$

The polynomial

$$l_n(x, a) = k_n(x, a)/k_n(a, a)$$

satisfies the following properties [6, p. 102]:

(i) $l_n(a, a) = 1$

(ii) $|l_n(x, a)| \leqslant 4/n\,|x-a|\; n \geqslant 3,\, x, a \in [-1, 1]$

(iii) $|l_n(x, a)| \leqslant 4\; n \geqslant 3,\, x, a \in [-1, 1]$.

The kernel polynomial $l_n(x, a) = \sum_{k=0}^{\prime n} T_k(x)\, T_k(a) / \sum_{k=0}^{\prime n} T_k^2(a)$ with $a = 0.5$ and $n = 20$ is plotted in Fig. 1.

The Lebesgue theorem insures that

$$\lim_{n \to \infty} \int_{-1}^{1} l_n(x, a)\, w(x)\, dx = 0$$

and so

$$\lim_{n \to \infty} c(l_n(x, a)) = \lim_{n \to \infty} \left[\int_{-1}^{1} l_n(x, a)\, w(x)\, dx + \sum_{i=1}^{q} A_i l_n(a_i, a) \right]$$

$$= \lim_{n \to \infty} \sum_{i=1}^{q} A_i l_n(a_i, a).$$

If $a = a_l$ and a_i distincts $\in [-1, 1]$ then

$$\lim_{n \to \infty} l_n(a_i, a_l) = 0 \qquad i \neq l$$

$$l_n(a_l, a_l) = 1$$

and thus we get

$$A_l = \lim_{n \to \infty} c[l_n(x, a_l)]$$

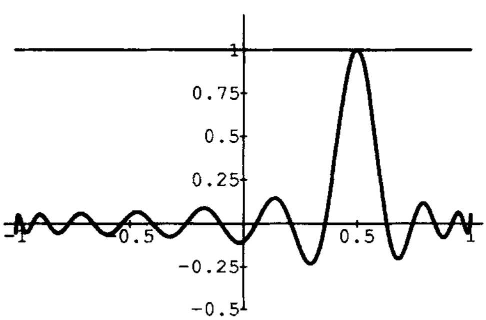

FIGURE 1

and the

THEOREM 1. *Let c and ω be two distributions satisfying* (1). *If ω satisfies*

$$\langle \omega, p \rangle = \int_{-1}^{1} p(x)\, w(x)\, dx \qquad w \in L^1$$

then

$$1 \leqslant l \leqslant q \qquad A_l = \lim_{n \to \infty} \frac{\sum_{k=0}^{\prime n} c(T_k)\, T_k(a_l)}{\sum_{k=0}^{\prime n} T_k^2(a_l)}$$

$$= \lim_{n \to \infty} \frac{4 \sum_{k=0}^{\prime n} c(T_k)\, T_k(a_l)}{(U_{2n}(a_l) + 2n + 1)}. \qquad (4)$$

See Section 4 for numerical examples.

Remark. If we do not know that the distribution has Dirac masses, a convenient way is to plot the function

$$c(l_n(x, a)) := \frac{\sum_{k=0}^{\prime n} c(T_k)\, T_k(a)}{\sum_{k=0}^{\prime n} T_k^2(a)} \qquad \text{for} \quad a \in [-1, 1].$$

If c has a Dirac mass at the point a then $\lim_{n \to \infty} c(l_n(x, a)) \neq 0$ (0 otherwise). In Fig. 2 we have plotted the function $c(l_n(x, a))$ (c acts on x), for $a \in [-1, 1]$, $c = \chi_{[-1,1]}\, dx + \delta_{0.5}$ and $n = 20$.

A similar result holds for the Christoffel function: if $P_k(c, x)$ is the orthogonal polynomial with respect to the linear functional c defined as in the introduction, then the Christoffel function is

$$\lambda_n(x) := \frac{1}{\sum_{k=0}^{n} P_k^2(c, x)}.$$

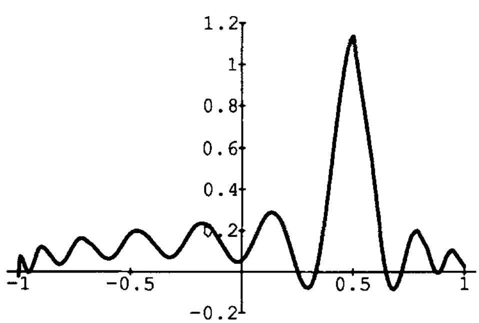

FIGURE 2

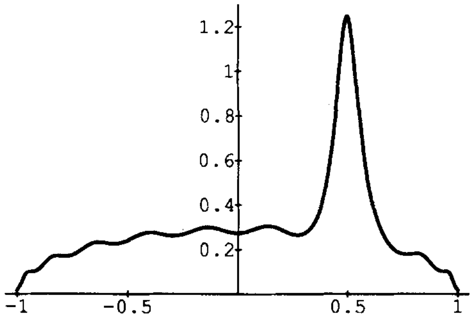

FIGURE 3

The values of $\lambda_n(x)$ can be computed using the Fortran routines published in [14] or with ε-algorithm on a computer algebra (see Section 3).

If the distribution c possesses a Dirac mass at the point a then this function will have a peak at this point a. In Fig. 3 is plotted the Christoffel function $\lambda_n(a)$, $a \in [-1, 1]$ for the distribution c, using the same number of moments as in the Fig. 2, that is to say for $n = 10$. Note that the Christoffel function does not require the knowledge of the support of the the distribution c [13].

Generally, computing modified moments from the ordinary power moments may lead to numerical instability [7] and the former are computed, if possible, directly from the expression of c [4, 8]. Here we have to compute modified moments for Chebyshev polynomials and it can be done by using their three-terms recurrence relationship.

3. Padé Approximants and ε-Algorithm

We assume now that c and ω are related by (5)

$$c = \omega + A\delta_a \qquad a \in \mathbf{R}, \qquad A \in \mathbf{C}, \tag{5}$$

and that ω satisfies

$$\langle \omega, p \rangle = \int_{\mathbf{R}} p(x)\, d\alpha(x) \qquad p \in P,$$

$$\alpha \text{ bounded, nondecreasing function.} \tag{6}$$

Relation (5) is equivalent to

$$(x-a)\,c=(x-a)\,\omega.$$

Let us set

$$\tilde{c}=(x-a)\,c \qquad \text{and} \qquad \tilde{\omega}=(x-a)\,\omega;$$

the moments $\tilde{c}_n$ and $\tilde{\omega}_n$ satisfy

$$\tilde{c}_n=c_{n+1}-ac_n=\tilde{\omega}_n=\omega_{n+1}-a\omega_n \qquad n=0, 1, \ldots .$$

The orthogonal polynomials with respect to $\tilde{c}=\tilde{\omega}$ will be denoted by $P_k(\tilde{c}, x)$ or $P_k(\tilde{\omega}, x)$. From the Christoffel Darboux identity, it arises

$$P_n(\tilde{c}, x)=\sum_{k=0}^{n} P_k(c, x)\,P_k(c, a)=P_k(\tilde{\omega}, x)=\sum_{k=0}^{n} P_k(\omega, x)\,P_k(\omega, a),$$

where $P_k(d, x)$ are orthonormal with respect to the functional d:

$$d(P_k(d, x)\,P_j(d, x))=\delta_{kj} \qquad k, j\in \mathbf{N}.$$

Let us now consider the sequence of polynomials:

$$l_n(x, a):=P_n(\tilde{c}, x)/P_n(\tilde{c}, a) \tag{7}$$

$$=\sum_{k=0}^{n} P_k(\omega, x)\,P_k(\omega, a)\Big/\sum_{k=0}^{n} P_k^2(\omega, a) \tag{8}$$

$$=\sum_{k=0}^{n} P_k(c, x)\,P_k(c, a)\Big/\sum_{k=0}^{n} P_k^2(c, a). \tag{9}$$

The moments $c(l_n(x, a))$ satisfy

$$c(l_n(x, a))=c\left(\frac{\sum_{k=0}^{n} P_k(c, x)\,P_k(c, a)}{\sum_{k=0}^{n} P_k^2(c, a)}\right)=\frac{1}{\sum_{k=0}^{n} P_k^2(c, a)}.$$

From relation (5), the moments $c(l_n(x, a))$ can also be expressed as

$$c(l_n(x, a))=\omega(l_n(x, a))+Al_n(a, a)$$

$$=\frac{1}{\sum_{k=0}^{n} P_k^2(\omega, a)}+A \qquad \text{(From (8))}.$$

A condition which insures that

$$\lim_{n \to \infty} 1 \Big/ \sum_{k=0}^{n} P_k^2(\omega, a) = 0$$

is that α be continuous at the point α and the distribution $d\alpha$ belongs to the set E of distributions uniquely determined by their moments [6, p. 62].

THEOREM 2. *If c and ω satisfy the relation* (5). *If ω satisfies* (6) *with $d\alpha \in E$ and α continuous at the point $a \in \mathbf{R}$.*
Then

$$A = \lim_{n \to \infty} c[l_n(x, a)] = \lim_{n \to \infty} \frac{1}{\sum_{k=0}^{n} P_k^2(c, a)}. \tag{10}$$

Remark. If $\alpha(x)$ has some jumps at the points $a_1, ..., a_q$ of magnitude $A_1, ..., A_q \in \mathbf{R}^+$ then the relation (5) becomes

$$c = \bar{\omega} + \sum_{i=1}^{q} A_i \, \delta a_i + A \delta_a$$

which is equivalent to the relation (1).

The quantities involved in (10) which are also required in the computation of the weights in Gauss–Christoffel quadrature formula can be evaluated from the three-terms relation satisfied by the orthogonal polynomials $P_k(c, a)$ (see [9, 8]).

These quantities $1/\sum P_k^2(c, a)$ can also be computed with ε-algorithm as explained in the following proposition:

PROPOSITION 1.

$$c[l_n(x, a)] = 1 \Big/ \sum_{k=0}^{n} P_k^2(c, a)$$

$$= \varepsilon_{2n}^{(0)},$$

where the quantities $\varepsilon_{2n}^{(0)}$ are computed with the ε-algorithm of Wynn,

$$\varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n+1)} + \frac{1}{\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)}}, \qquad k, n = 0, 1, 2, ...,$$

with the initial conditions

$$\varepsilon_{-1}^{(n)} = 0 \qquad n = 0, 1, 2, \ldots$$

$$\varepsilon_0^{(n)} = c_n/a^n, \qquad a \neq 0$$

$$= (I+E)^n c_0 := \sum_{k=0}^{n} \binom{n}{k} c_k, \qquad a = 0.$$

Proof. First, we write $c[l_n(x, a)]$ in terms of Padé Approximants. From

$$\tilde{c} = (x-a)\, c$$

we get

$$\tilde{c}_n = c_{n+1} - ac_n \qquad n = 0, 1, 2, \ldots,$$

and so

$$c_n = \tilde{c}_{n-1} + a\tilde{c}_{n-2} + a^2\tilde{c}_{n-3} + \cdots + a^{n-1}\tilde{c}_0 + a^n c_0.$$

Thus

$$c(p) = \tilde{c}\left(\frac{p(x)-p(a)}{x-a}\right) + c_0 p(a) \qquad p \in P.$$

Applying the functional c to equality (7) gives

$$c[l_n(x, a)] = \frac{c[P_n(\tilde{c}, x)]}{P_n(\tilde{c}, a)}$$

$$= \left(\tilde{c}\left[\frac{P_n(\tilde{c}, x) - P_n(\tilde{c}, a)}{x-a}\right] + c_0 P_n(\tilde{c}, a)\right) \Big/ P_n(\tilde{c}, a)$$

$$\Rightarrow c[l_n(x, a)] = c_0 + a^{-1}[n-1/n]_f(a^{-1})$$

$$= [n/n]_f(a^{-1}), \qquad [5, \text{Chap. } 3],$$

where $[n/n]$ is the Padé Approximant to the function

$$f(t) := c_0 + \tilde{c}_0 t + \tilde{c}_1 t^2 + \cdots.$$

Now, it is well known that Padé approximant to the series f and ε-algorithm are related by [5, p. 159]

$$[n/n]_f(t) = \varepsilon_{2n}^{(0)},$$

where ε-algorithm is applied to the sequence of partial sums of $f: f_n(t) = c_0 + \tilde{c}_0 t + \tilde{c}_1 t^2 + \cdots + \tilde{c}_{n-1} t^n$.

Here $f_n(1/a) = c_0 + \tilde{c}_0/a + \tilde{c}_1/a^2 + \cdots + \tilde{c}_{n-1}/a^n = c_n/a^n$ if $a \in R^*$.

For the general case, $a \in R$, we can use the homographic invariance under argument transformation:[1]

Define an homographic transformation of the argument t,

$$u = \frac{t}{1+bt} \leftrightarrow t = \frac{u}{1-bu},$$

and thereby a new function $g(u) = f(t) = f(u/(1-bu))$.

Then (Theorem of Baker, Gammel, and Wills, see [3, p. 32, t. 1])

$$[n/n]_f(t) = [n/n]_g(u).$$

From

$$f(t) = c_0 + \tilde{c}_0 t + \tilde{c}_1 t^2 + \cdots,$$

$$g(u) = c_0 + \tilde{c}_0 \left(\frac{u}{1-bu}\right) + \tilde{c}_1 \left(\frac{u}{1-bu}\right)^2 + \cdots$$

$$= c_0 + \tilde{c}_0 u + (\tilde{c}_0 b + \tilde{c}_1) u^2 + \cdots.$$

The partial sums of the series f at $t = 1/a$ corresponding to the partial sums of g at $u = 1/(a+b)$ are

$$\varepsilon_0^{(n)} = c_0 + (c_1 - ac_0)\frac{1}{a+b} + [(c_1 - ac_0)b + (c_2 - ac_1)]\frac{1}{(a+b)^2} + \cdots$$

$$= \frac{(b+E)^n c_0}{(b+a)^n} = \frac{\sum_{j=0}^{n} \binom{n}{j} b^{n-j} c_j}{(b+a)^n},$$

and if $a = 0$ then we can take $b = 1$ and so $\varepsilon_0^{(n)} = (I+E)^n c_0$. ∎

4. Numerical Examples

Example 1.

$$c = dx + \delta_{0.5} \qquad \text{on } [-1, 1].$$

[1] I thank A. Magnus for this proof.

Only the moments and the support of c are assumed to be given. Since the distribution dx is positive, ε-algorithm can be used as well as the Chebyshev polynomials. The mass at the point $a=0.5$ is $A=1$ (See Table I),

$$c_n = \frac{1-(-1)^n}{n+1} + 0.5^n \qquad \text{column (2)}$$

$$c[l_n(x,a)] = \sum_{k=0}^{n}{}' c(T_k)\, T_k(a) \Big/ \sum_{k=0}^{n}{}' T_k^2(a) \qquad \text{column (4)}$$

$$\varepsilon_{2n}^{(0)}(c_n a^{-n}) \qquad \text{column (5)}$$

($c(T_n)$ (column (3)) does not converge to 0, which confirms the presence of Dirac masses; see [16]).

We can see that both $c(l_n(x,a))$ and $\varepsilon_{2n}^{(0)}(c_n a^{-n})$ converge to $A=1$. We can remark that the convergence of the first one is faster than that of the second one, but it must be noted that ε-algorithm does not require the knowledge of the support of c. In Table I the $\varepsilon_n^{(0)}$ with n odd are omitted because they are only used for the computation of the ε-array (see Proposition 1).

TABLE I

Computation of the Mass A at the Point $a=0.5$ for $c=\chi_{[-1,1]}\,dx+\delta_{0.5}$

(1) n	(2) c_n	(3) $c(T_n)$	(4) $c(l_n)$	(5) $\varepsilon_n^{(0)}$
0	3.0000	1.5000	3.0000	3.0000
1	0.5000	0.5000	2.3330	
2	0.9167	−1.1667	2.3330	2.1429
3	0.1250	−1.0000	1.6670	
4	0.4625	−0.6333	1.6220	2.0940
5	0.0313	0.5000	1.5600	
6	0.3013	0.9429	1.3840	1.6313
7	0.0078	0.5000	1.3580	
8	0.2261	−0.5317	1.3400	1.5102
9	0.0020	−1.0000	1.2720	
10	0.1828	−0.5202	1.2610	1.4989
15	0.0000	−1.0000	1.1700	
20	0.0952	−0.5050	1.1360	1.2406
25	0.0000	0.5000	1.1070	
30	0.0645	0.9978	1.0880	1.1670
35	0.0000	0.5000	1.0780	
40	0.0488	−0.5008	1.0670	1.1332

EXAMPLE 2.

$$c = \omega + 4\delta_{7.5} \qquad \text{where} \quad \langle \omega, p \rangle = \int_0^{\infty} p(x)\, e^{-x}\, dx.$$

Here, the method of Section 2 cannot be used, since the interval is infinite. By applying the ε-algorithm (see Section 3) to the sequence $c_n\ 7.5^{-n}$, the limit of $\varepsilon_{2n}^{(0)}$ is $A = 4$,

$$c_n = n! + 4 \times 7.5^n \Rightarrow c_n 7.5^{-n} = n!\ 7{,}5^{-n} + 4$$

(See Table II).

EXAMPLE 3. Let the distribution be defined as

$$c = \omega \qquad \text{with} \quad \omega(x^n) = \int_{-1}^{1} x^n w(x)\, dx,$$

where

$$w(x) = x + 1 \qquad \text{on} \quad [-1, 0.5[$$
$$w(x) = 1 - x \qquad \text{on} \quad]0.5, 1].$$

The coefficients

$$c_n = \frac{0.5^{n+1}}{n+2} + \frac{2}{(n+1)(n+2)} \qquad n \text{ even}$$
$$c_n = 0.5^{n+1}/(n+2) \qquad n \text{ odd}$$

are supposed to be given. The goal is, here, to compute the jump of w at the point $a = 0.5$ (this value $a = 0.5$ can be approximated by the method explained in [16]).

TABLE II

Computation of the Mass A at the Point $a = 7.5$ for $c = e^{-x}\, dx + 4\delta_{7.5}$

n	$\varepsilon_n^{(0)}$	n	$\varepsilon_n^{(0)}$	n	$\varepsilon_n^{(0)}$	n	$\varepsilon_n^{(0)}$
0	5.000000	12	4.002010	24	4.001458	36	4.001123
2	4.023121	14	4.001910	26	4.001338	38	4.001105
4	4.004119	16	4.001858	28	4.001284	40	4.001105
6	4.003355	18	4.001635	30	4.001283	42	4.001083
8	4.002554	20	4.001532	32	4.001254	44	4.001036
10	4.002495	22	4.001532	34	4.001182	46	4.000993

The jumps of w will appear in the derivative (in the sense of distributions) of w. If w has a finite number of jumps of finite magnitude A_i at points $a_1, ..., a_q$ then the derivative of w satisfies

$$w' = \{w\}' + \sum_{i=1}^{q} A_i \delta_{a_i},$$

where $\{w\}'$ is the derivative of w in the usual sense, if it exists.

The moments of w' can be computed as

$$\forall p \in P, \qquad \int_{-1}^{1} p(x)\, w'(x)\, dx = -\int_{-1}^{1} p'(x)\, w(x)\, dx$$

by integration by parts, and so

$$c'_n := \langle c', x^n \rangle = -nc_{n-1} \qquad n \geqslant 1$$

$$c'_0 = 0.$$

c' is not a positive distribution, so the ε-algorithm is uneffective. We can only use the Chebyshev kernel (See Table III). The sequence $c'(T_n)$ does

TABLE III

Computation of the Jump of $w(x) = 1 + x$, $x \in [-1, 0.5]$, $1 - x$, $x \in [0.5, 1]$ at the Point $a = 0.5$

(1) n	(2) c'_n	(3) $c'(T_n)$	(4) $c'(l_n(x, a))$
0	0.000	0.000	0.00000
1	−1.250	−1.250	−0.83333
2	−0.167	−0.333	−0.45833
3	−0.594	1.375	−0.91667
4	−0.050	0.933	−1.02222
5	−0.359	−0.125	−0.94500
6	−0.013	−1.029	−0.96888
7	−0.257	−0.688	−0.99595
8	−0.003	0.317	−0.97339
9	−0.202	1.037	−0.98621
10	−0.001	0.657	−1.00178
15	−0.125	1.013	−0.99447
20	−0.000	0.426	−0.99605
25	−0.077	−0.558	−0.99988
30	−0.000	−1.001	−0.99842
35	−0.056	−0.456	−0.99879
40	−0.000	−0.538	−1.00004

not converge to 0, which indicates that the distribution c' has Dirac masses. The point $a=0.5$ can be calculated with

$$a=\lim_{n\to\infty}[c'(T_{n+1})+c'(T_{n-1})]/2c'(T_n) \qquad \text{(see [16])}$$

and the mass $A=-1$ satisfies

$$A=\lim_{n\to\infty} c'(l_n(x,a))$$

5. Limit of Totally Monotonic Sequences

An important application of the previous sections is the calculation of the limit of totally monotonic sequences.

Definition. A sequence $(c_n)_{n\in\mathbb{N}}$ is said to be totally monotonic ($c_n\in \text{TM}$) if

$$(-1)^k\Delta^k c_n\geq 0 \qquad \text{for} \quad n,k=0,1,...$$

or equivalently if there exists a nondecreasing function α such that

$$c_n=\int_0^1 x^n\,d\alpha(x) \qquad n=0,1,....$$

It is well known that a TM sequence is always convergent and that the limit l satisfies

$$l=\alpha(1)-\alpha(1^-) \qquad [5, \text{pp. } 116\text{–}120].$$

Since α is nondecreasing, we can apply Theorem 2 and thus the limit of the sequence $(c_n)_n$ can be found by the ε-algorithm [5, p. 165]. It is possible to generalize the result of Section 2 for Jacobi polynomials on [0, 1].

In the particular case where $a=1$, it is possible to extend the properties of the Chebyshev reproducing kernel $l_n(x,a)$ of Section 2 to Jacobi polynomials on [0, 1] (shifted Jacobi polynomials).

Lemma 1. *Let $P_n^{*(\alpha,\beta)}(x)$ be the shifted Jacobi polynomial with $\beta>-1$, $\alpha>-1$. Let $l_n(x)=P_n^{*(\alpha,\beta)}(x)/P_n^{*(\alpha,\beta)}(1)$ then*

(i) $l_n(1)=1$

(ii) $\beta<\alpha,\ \alpha>-\frac{1}{2}\Rightarrow \lim_{n\to\infty} l_n(x)=0,\ \forall x\in[0,1[$

(iii) $|l_n(x)|\leqslant 1\ \forall x\in[0,1]$.

The polynomial $l_n(x)$ for $\alpha = 1$, $\beta = 0$ and $n = 15$ is plotted in Fig. 4.

Proof. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy [18, p. 58]

$$\int_{-1}^{1} P_n^{(\alpha,\beta)} P_m^{(\alpha,\beta)}(x)(1-x)^\alpha (1+x)^\beta \, dx = 0 \qquad \text{if} \quad n \neq m.$$

The normalization of $P_n^{(\alpha,\beta)}$ is taken such that

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

The shifted Jacobi polynomials are defined by

$$P_n^{*(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1).$$

$P_n^{*(\alpha,\beta)}$ satisfies

$$\int_0^1 P_n^{*(\alpha,\beta)}(x) \, P_n^{*(\alpha,\beta)}(x)(1-x)^\alpha x^\beta \, dx = 0 \qquad \text{if} \quad n \neq m.$$

$$P_n^{*(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$$

$$P_n^{*(\alpha,\beta)}(0) = P_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{n+\beta}{n}.$$

Moreover, if $\beta \langle \alpha, \alpha \rangle -\frac{1}{2}$,

$$\max_{0 \leqslant x \leqslant 1} |P_n^{*(\alpha,\beta)}(x)| = \binom{n+\alpha}{n} \approx n^\alpha, \qquad \text{if} \quad \alpha > -\tfrac{1}{2},$$

and

$$|P_n^{*(\alpha,\beta)}(x)| = 0(n^{-1/2}), \qquad x \in [0, 1[\qquad [18, \text{p. } 168]$$

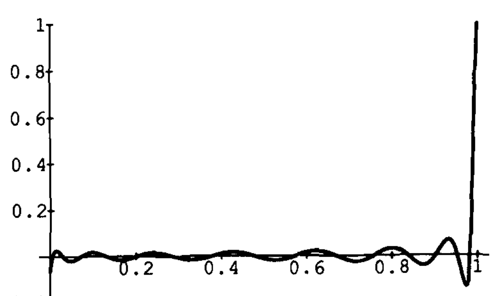

FIGURE 4

Thus

(i) $l_n(1) = P_n^{*(\alpha,\beta)}(1)/P_n^{*(\alpha,\beta)}(1) = 1$

(ii) $|l_n(x)| = P_n^{*(\alpha,\beta)}(x)/\binom{n+\alpha}{n} \leqslant 1$, $\forall x \in [0, 1]$

(iii) $|l_n(x)| = 0(n^{-1/2})/\binom{n+\alpha}{n} \to 0$ when $n \to \infty$ for $\alpha > -\frac{1}{2}$, $\forall x \in [0, 1[$. ∎

The Lebesgue theorem insures that

$$\lim_{n\to\infty} c(l_n(x)) = \lim_{n\to\infty} \int_0^1 l_n(x)\, d\alpha(x) = l.$$

THEOREM 3. *If the sequence* $(c_n)_{n\in\mathbf{N}} \in TM$ *and* $\alpha > -\frac{1}{2}$, $\alpha > \beta$ *then*

$$\lim_{n\to\infty} \frac{c(P_n^{*(\alpha,\beta)}(x))}{P_n^{*(\alpha,\beta)}(1)} = \lim_{n\to\infty} c_n.$$

TABLE IV

Computation (Carried Out with 30 Digits) of the Limit of the Sequence $c_n = 1/(n+1) + \sum_{k=2}^{n+2} [1-(1-(1/k^3)^k]$

(1) n	(2) c_n	(3) $\varepsilon_n^{(0)}(c_n)$	(4) $c(l_n)$	(5) $\varepsilon_n^{(0)}(c(l_n))$
0	1.234	1.23437	1.23437500000	1.23437500000000000000
1	0.841		0.64494503506	
2	0.735	0.69698	0.62035175592	0.61928095192382953370
3	0.691		0.62250209505	
4	0.669	0.64161	0.62228637748	0.62230594922597273668
5	0.656		0.62231515838	
6	0.647	0.62938	0.62231051757	0.62231120734183040435
7	0.642		0.62231137699	
8	0.638	0.62548	0.62231117901	0.62231122482714539152
9	0.635		0.62231125343	
10	0.633	0.62394	0.62231120596	0.62231122673275067693
11	0.631		0.62231124333	
12	0.630	0.62323	0.62231121309	0.62231122657072932174
13	0.629		0.62231123724	
14	0.628	0.62287	0.62231121822	0.62231122657169596626
15	0.627		0.62231123308	
16	0.627	0.62267	0.62231122151	0.62231122657176348050
17	0.626		0.62231123052	
18	0.626	0.6225	0.62231122348	0.62231122657176407405
19	0.625		0.62231122900	
20	0.625	0.62247	0.62231122465	0.62231122657176411017

The computation of $c(P_n^{*(\alpha,\beta)}(x))$ is very easy with the following expression of $P_n^{*(\alpha,\beta)}(x)$ [18, p. 68],

$$P_n^{*(\alpha,\beta)}(x)=\sum_{\nu=0}^{n}\binom{n+\alpha}{n-\nu}\binom{n+\beta}{\nu}(x-1)^{\nu}x^{n-\nu},$$

and

$$c(P_n^{*(\alpha,\beta)}(x))=\sum_{\nu=0}^{n}\binom{n+\alpha}{n-\nu}\binom{n+\beta}{\nu}c((x-1)^{\nu}x^{n-\nu})$$
$$=\sum_{\nu=0}^{n}\binom{n+\alpha}{n-\nu}\binom{n+\beta}{\nu}\Delta^{\nu}c_{n-\nu}.$$

NUMERICAL EXAMPLE.

$$c_n=1/(n+1)+\sum_{k=2}^{n+2}\left[1-\left(1-\frac{1}{k^3}\right)^k\right]\qquad n\geqslant 0.$$

The sequence $(c_n)_n$ is totally monotonic and the convergence is logarithmic.

Column (3) contains the diagonal of the ε-array, for the sequence $(c_n)_n$. Column (4) contains $c(l_n(x))$ for $\alpha=1$, $\beta=0$. Using exact arithmetic, we saw that $c(l_n)$ is totally oscillating around its limit l up to 50 that is $((-1)^n(c(l_n(x))-l))\in \mathrm{TM}$, and applying the ε-algorithm to it gives column (5) (see Table IV). The limit seems to be: 0.622311226571764110266.... (computed by means of E-algorithm with auxiliary sequences $1/(n+1)$, $1/(n+1)^2$, $1/(n+1)^3$, ... (See [12, 17])).

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